
Contact intensity and extended hydrodynamics in the BCS-BEC crossover

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Summary. In the first part of this chapter we analyze the contact intensity C , which has been introduced by Tan [Ann. Phys. **323**, 2952 (2008)] and appears in several physical observables of the strongly correlated two-component Fermi gas. We calculate the contact C in the full BCS-BEC crossover for a uniform superfluid Fermi gas by using an efficient parametrization of the ground-state energy. In the case of harmonic confinement, within the Thomas-Fermi approximation, we derive analytical formulas of C in the three relevant limits of the crossover. In the second part of this chapter we discuss the extended superfluid hydrodynamics we have recently proposed to describe static and dynamical collective properties of the Fermi gas in the BCS-BEC crossover. In particular we show the relation with the effective theory for the Goldstone field derived by Son and Wingate [Ann. Phys. **321**, 197 (2006)] on the basis of conformal invariance. By using our equations of extended hydrodynamics we determine nonlinear sound waves, static response function and structure factor of a generic superfluid at zero temperature.

1 Contact intensity

It has been proved by Tan [1] that the momentum distribution $\rho_\sigma(k)$ in an arbitrary system consisting of fermions in two spin states ($\sigma = \uparrow, \downarrow$) with a large scattering length has a tail that falls off as

$$\rho_\sigma(k) \sim \frac{C}{k^4} \quad (1)$$

for $k \rightarrow \infty$, where C is the so-called contact intensity [1]. Here large s-wave scattering length a means that $|a| \gg r_0$, where r_0 is the effective interaction radius. Under this condition Tan [1] has shown that C is related to the total energy E of the Fermi system by the rigorous expression

$$C = \frac{4\pi m a^2}{\hbar^2} \frac{dE}{da}, \quad (2)$$

where the derivative is taken under constant entropy and, in general, C depends on the number N of fermions, the scattering length a and the parameters of the

trapping potential [2, 3]. Remarkably, Eqs. (1) and (2) work also at finite temperature and in this case C will be a function of T [2, 4]. Tan has also derived, for finite scattering lengths, a generalized virial theorem and a generalized pressure relation where the contact C appears [3]. The contact intensity C appears also in other physical observables of the strongly correlated Fermi system. For instance, the radio-frequency spectroscopy shift is proportional to C [5, 6, 7, 8], and the same happens to the photoassociation rate [9]. Very recently, it has been shown that the contact C gives the asymptotic tail behavior of the shear viscosity as a function of the frequency [10].

Using the methods of quantum field theory, Braaten and Platter have rederived [11, 12] the Tan's universal relations [1, 2, 3]. In addition, they have shown that the contact intensity can be written as

$$C = \int g^2 \langle \hat{\psi}_\uparrow^\dagger(\mathbf{r}) \hat{\psi}_\downarrow^\dagger(\mathbf{r}) \hat{\psi}_\downarrow(\mathbf{r}) \hat{\psi}_\uparrow(\mathbf{r}) \rangle d^3\mathbf{r}, \quad (3)$$

where $\hat{\psi}_\sigma(\mathbf{r})$ is the fermionic field operator of spin σ and $g = 4\pi a/(1 - (2ak_{cut}/\pi))$ is the bare coupling constant of the Fermi pseudo-potential interaction, with k_{cut} the ultraviolet wavenumber cutoff [11, 13]. Braaten and Platter have also shown that the number $\mathcal{N}_{pair}(\mathbf{r})$ of pairs of fermions with opposite spins in a small ball of volume $4\pi s^3/3$ centered at the point \mathbf{r} scales as $s^4 \mathcal{C}(\mathbf{r})/2$ for $s \rightarrow 0$, where $\mathcal{C}(\mathbf{r}) = g^2 \langle \hat{\psi}_\uparrow^\dagger(\mathbf{r}) \hat{\psi}_\downarrow^\dagger(\mathbf{r}) \hat{\psi}_\downarrow(\mathbf{r}) \hat{\psi}_\uparrow(\mathbf{r}) \rangle$ is the contact density [11, 13].

Explicit expressions of the universal quantity C have been derived by Tan [1] for a uniform superfluid Fermi gas at zero temperature only in three limits: the Bardeen-Cooper-Schrieffer (BCS) limit of weakly bound Cooper pairs, the unitarity limit of infinite scattering length, and the Bose-Einstein condensate (BEC) limit of weakly-interacting bosonic molecular pairs.

In this section we calculate the contact C as a function of the inverse scattering parameter $1/(k_F a)$ for a uniform superfluid Fermi gas in the full BCS-BEC crossover. To perform this calculation we use an efficient fitting formula of the ground-state energy [25, 26] and the Tan's equation (2). We find that the contact C has a maximum close to the unitarity limit of infinite scattering length. We also consider the interacting Fermi system under harmonic confinement. For this superfluid Fermi cloud we derive analytical formulas of the contact C in the three relevant limits of the crossover.

1.1 Uniform superfluid Fermi gas at zero temperature

In the case of a zero-temperature uniform two-component superfluid Fermi gas of total density $n = n_\uparrow + n_\downarrow$ ($n_\uparrow = n_\downarrow$), large scattering length a ($a \gg r_0$) in a volume V , the energy density can be written as

$$\frac{E}{V} = \frac{3}{5} n \epsilon_F f\left(\frac{1}{k_F a}\right), \quad (4)$$

where $f(y)$ is a universal function of inverse interaction parameter $y = 1/(k_F a)$, with $\epsilon_F = \hbar^2 k_F^2/(2m)$ the Fermi energy and $k_F = (3\pi^2 n)^{1/3}$ the Fermi wave number [4]. We observe that at finite temperature T the function $f(y)$ is substituted by a more general universal function $\Phi(y, t)$, where $t = T/\epsilon_F$ is the scaled temperature.

$\Phi(y, t)$ has been studied with Monte Carlo methods by Bulgac, Drut, and Migierski [14, 15], but only in the unitarity limit ($y = 0$).

It is straightforward to derive from Eq. (2) the expression of the contact density

$$\frac{C}{V} = -\frac{6\pi}{5} k_F n \frac{df}{dy}. \quad (5)$$

The behavior of $f(y)$ is well known in three relevant limits:

$$f(y) = \begin{cases} 1 + \frac{10}{9\pi} \frac{1}{y} + O(1/y^2), & y \ll -1 \\ \xi - \zeta y + O(y^2), & -1 \ll y \ll 1 \\ \frac{5\mathcal{P}}{18\pi} \frac{1}{y} + O(1/y^{5/2}), & y \gg 1 \end{cases} \quad (6)$$

In fact, in the weakly attractive limit ($y \ll -1$) one expects a BCS Fermi gas of weakly bound Cooper pairs. As the superfluid gap correction is exponentially small, the function $f(y)$ follows the Fermi-gas expansion [16, 17]. In the so-called unitarity limit ($y = 0$) one expects that the energy per particle is proportional to that of a non-interacting Fermi gas with the universal constant ξ given by $\xi \simeq 0.42$ [18]. Note that more recent auxiliary-field Monte Carlo results [19] predict a smaller value for ξ , namely $\xi \simeq 0.38$, while the experiment performed at Ecole Normale Supérieure [20] suggests $\xi \simeq 0.40$. The first correction to this behavior, shown in Eq. (6), has been estimated from Monte Carlo data with $\zeta \simeq 1$ [21]. In the weak BEC limit ($y \gg 1$) one expects a weakly repulsive Bose gas of dimers. Such Bose-condensed molecules of mass $m_M = 2m$ and density $n_M = n/2$ interact with a positive scattering length $a_M = \mathcal{P}a$ with $\mathcal{P} \simeq 0.6$, as predicted by Petrov *et al.* [22]. In this regime, after subtraction of the molecular binding energy, the function $f(y)$ follows the Bose-gas expansion [23]. It is easy to obtain the contact intensity C by using Eqs. (2) and (6) in the relevant limits of the crossover. One finds

$$\frac{C}{V} = \begin{cases} \frac{4}{3} k_F^3 n a^2 + O(a^3), & y \ll -1 \\ \frac{6\pi}{5} k_F n \zeta + O(1/a), & -1 \ll y \ll 1 \\ \frac{\mathcal{P}}{3} k_F^3 n a^2 + O(a^{7/2}), & y \gg 1 \end{cases} \quad (7)$$

in agreement with the previous determinations of Tan [1] and Werner, Tarruell and Castin [9]. Notice that in the BEC limit we have removed the binding energy of molecules. Moreover, very recently finite temperature corrections to Eq. (7) were given in Ref. [24].

1.2 Contact intensity in the BCS-BEC crossover

Now we want to calculate the behavior of C in the full BCS-BEC crossover. In 2005 we have proposed [25] the following analytical fitting formula

$$f(y) = \alpha_1 - \alpha_2 \arctan \left(\alpha_3 y \frac{\beta_1 + |y|}{\beta_2 + |y|} \right), \quad (8)$$

interpolating the Monte Carlo energy per particle [18] and the limiting behaviors for large and small $|y|$. Eq. (8) is very reliable [25, 26] and it has been successfully used by various authors for studying ground-state and collective properties of this superfluid Fermi system [27, 28, 29, 30, 31]. The parameter α_1 is fixed by the value ξ of $f(y)$ at $y = 0$, the parameter α_2 is fixed by the value of $f(y)$ at $y = \infty$, and

α_3 is fixed by the asymptotic $1/y$ coefficient of $\epsilon(y)$ at large $|y|$. The ratio β_1/β_2 is determined by the linear behavior ζ of $\epsilon(y)$ near $y = 0$. The value of β_1 is then determined by minimizing the mean square deviation from the Monte-Carlo data. Of course, we have considered two different set of parameters: one set in the BCS region ($y < 0$) and a separate set in the BEC region ($y > 0$) [25]. Table 1 of [25] reports the values of these parameters, with $\zeta = \zeta_- = 1$ in the BCS region but $\zeta = \zeta_+ = 1/3$ in the BEC region.

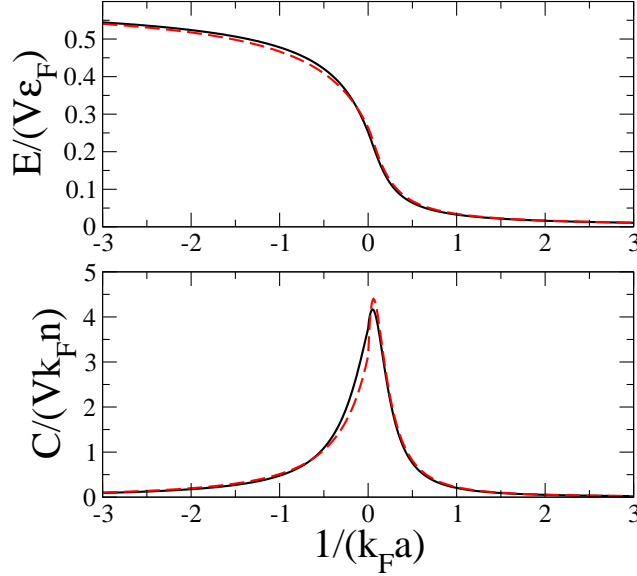


Fig. 1. Upper panel: Scaled ground-state energy $E/(V\epsilon_F)$ of the uniform Fermi gas as a function of the inverse interaction parameter $1/(k_F a)$ in the BCS-BEC crossover. Lower panel: Scaled contact intensity $C/(V k_F n)$ of the Fermi system as a function of the inverse interaction parameter $1/(k_F a)$. Here V and $n = N/V$ are the volume and the density of the Fermi gas, $k_F = (3\pi^2 n)^{1/3}$ is the Fermi wave number and $\epsilon_F = \hbar^2 k_F^2 / (2m)$ is the Fermi energy. Two different parametrization of the universal function $f(y)$: solid lines are obtained with Eq. (8), while dashed lines are calculated using the Pade approximant of [32].

Here we use Eq. (8) to calculate the contact density given by Eq. (5), but contrary to [25], we choose $\zeta_+ = \zeta_- = \zeta = 1$ to ensure the continuity of $f'(y)$ at $y = 0$. Notice that the recent experimental results obtained at Ecole Normale Supérieure [20] indeed suggest the continuity of $f'(y)$ at $y = 0$. In this way, in the BEC region $\beta_2 = 0.1517$ while β_1 is unchanged (see Table. 1 of [25]). In the upper panel of Fig. 1 we plot the ground-state energy E while in the lower panel we plot the contact C , both as a function of the inverse interaction parameter $y = 1/(k_F a)$. For comparison, in addition to the data obtained with our method (solid lines), we insert also the

results (dashed lines) one obtains using the Pade parametrization of $f(y)$ proposed by Kim and Zubarev [32]. The figure shows that solid and dashed lines are always close each other, apart for $-1 \ll y \leq 0$ where our fitting formula is smoother (and closer to the Monte Carlo data [18]). Moreover, the scaled contact $C/(V k_F n)$ as a function of $1/(k_F a)$ has its maximum near to the unitarity limit $1/(k_F a) = 0$: the position of the maximum is located at $1/(k_F a) \simeq 0.05$. Remarkably, the scaled contact has a behavior quite similar to the Landau's critical velocity v_c (at which there is the breaking of superfluid motion), calculated along the BCS-BEC crossover. In fact, also v_c goes to zero for $y \rightarrow \pm\infty$ and it has a peak at $y \simeq 0.08$ [33]. Clearly, the contact C exhibits a maximum close to unitarity because we have subtracted the molecular binding energy contribution, given by y^2 in the BEC side ($y > 0$). Including this energy term it is easy to show that C increases monotonically from the BCS side to the BEC side, according the definition in Eq. (2). Nevertheless, as previously stated, the radio-frequency spectroscopy shift is proportional to C , and its maximum around unitarity has been shown in Ref. [6], where the same artificial subtraction has been adopted.

1.3 Trapped superfluid Fermi gas

Let us now consider the superfluid Fermi gas under an external harmonic confinement

$$U(\mathbf{r}) = \frac{1}{2} m \omega^2 (x^2 + y^2 + z^2) . \quad (9)$$

In the limit of a large number N of fermions we can use the local density (Thomas-Fermi) approximation [34, 35, 36, 37, 38, 39, 40, 41] and the energy of the system can be written as

$$E = \int \left\{ \frac{3}{5} n(\mathbf{r}) \epsilon_F(\mathbf{r}) f\left(\frac{1}{k_F(\mathbf{r})a}\right) + n(\mathbf{r}) U(\mathbf{r}) \right\} d^3 \mathbf{r} , \quad (10)$$

where $\epsilon_F(\mathbf{r}) = \hbar^2 k_F(\mathbf{r})^2 / (2m)$ is the local Fermi energy and $k_F = (3\pi^2 n(\mathbf{r}))^{1/3}$ is the local Fermi wave number. As in the uniform case, this expression is not very useful without the knowledge of the universal function $f(y)$.

The numerical calculation of the contact indensity C for a harmonically trapped Fermi superfluid in the full BCS-BEC crossover by using Eqs. (2) and (8) is very demanding. Consequently, we calculate C only in the three relevant limits of the BEC-BEC crossover. In these limits we obtain useful analytical expressions for the contact C .

BCS limit. In the BCS limit ($a \rightarrow 0^-$) from Eqs. (6) and (10) we find

$$\frac{dE}{da} = \frac{3}{5} \frac{10}{9\pi} \int \{ \epsilon_F(\mathbf{r}) n(\mathbf{r}) k_F(\mathbf{r}) \} d^3 \mathbf{r} , \quad (11)$$

where $n(\mathbf{r})$ is the density profile of the ideal Fermi gas in the harmonic potential (9), given by [42]

$$n(\mathbf{r}) = \frac{2\sqrt{2}}{3\pi^2 a_H^3} (3N)^{1/2} \left(1 - \frac{r^2}{r_F^2}\right)^{3/2} , \quad (12)$$

where $r_F = \sqrt{2}(3N)^{1/6} a_H$ is the Fermi radius of the cloud, with $a_H = \sqrt{\hbar/(m\omega)}$ the characteristic harmonic length. Inserting this density profile into Eq. (11) and using Eq. (8) we get the contact intensity in the BCS limit:

$$C = \frac{4096\sqrt{2}}{2835\pi} \frac{1}{a_H} \left(\frac{a}{a_H} \right)^2 (3N)^{3/2}. \quad (13)$$

Unitarity limit. In the unitarity limit ($a \rightarrow \pm\infty$) from Eqs. (6) and (10) we have

$$\frac{dE}{da} = \frac{3}{5} \frac{\zeta}{a^2} \int \left\{ \epsilon_F(\mathbf{r}) n(\mathbf{r}) \frac{1}{k_F(\mathbf{r})} \right\} d^3\mathbf{r}, \quad (14)$$

where $n(\mathbf{r})$ is the density profile of the unitary Fermi gas in the potential (9), namely [34]

$$n(\mathbf{r}) = \frac{2\sqrt{2}}{3\pi^2 a_H^3 \xi^{3/4}} (3N)^{1/2} \left(1 - \frac{r^2}{r_F^2}\right)^{3/2}, \quad (15)$$

where $r_F = \sqrt{2}\xi^{1/4}(3N)^{1/6}a_H$ is the Fermi radius of the unitary cloud. Inserting this density profile into Eq. (14) and using Eq. (8) we obtain the contact in the unitarity limit:

$$C = \frac{512\sqrt{2}}{525} \frac{\zeta}{\xi^{1/4}} \frac{1}{a_H} (3N)^{7/6}. \quad (16)$$

BEC limit. In the BEC limit ($a \rightarrow 0^+$) it is straightforward to calculate the contact C . In fact, the explicit formula of the ground-state energy of the dilute BEC is well known [43] and for a BEC of molecules it is given by

$$E = \frac{5}{7} \frac{\hbar\omega}{2} \left(\frac{15\mathcal{P}a}{a_H} \right)^{2/5} \left(\frac{N}{2} \right)^{7/5}, \quad (17)$$

where $\mathcal{P} \simeq 0.6$ and $N/2$ is the number of molecules. Then from Eq. (2) the contact intensity reads

$$C = \frac{2\pi}{7} \frac{1}{a_H} \left(\frac{15\mathcal{P}}{2} \right)^{2/5} \left(\frac{aN}{a_H} \right)^{7/5}. \quad (18)$$

2 Extended superfluid hydrodynamics

In this section we discuss the extended Lagrangian density of superfluids we have proposed few years ago [34] and applied to study mainly the unitary Fermi gas [34, 35, 36, 37, 38, 39, 40, 41]. The internal energy density of this Lagrangian contains a term proportional to the kinetic energy of a uniform non interacting gas of fermions, plus a gradient correction of the von-Weizsacker form $\lambda\hbar^2/(8m)(\nabla n/n)^2$ [44]. This approach has been adopted for studying the quantum hydrodynamics of electrons by March and Tosi [45], and by Zaremba and Tso [46]. In the context of the BCS-BEC crossover, the gradient term is quite standard [25, 32, 47, 48, 49, 50, 51, 52, 53]. In particular we show the relation between our approach and the effective theory for the Goldstone field derived by Son and Wingate [54], and improved by Manes and Valle [55], on the basis of conformal invariance. Finally, by using our equations of extended superfluid hydrodynamics at zero temperature we calculate sound waves, static response function and structure factor of a generic superfluid.

The extended Lagrangian density of superfluids is given by

$$\mathcal{L} = -\hbar \dot{\theta} n - \frac{\hbar^2}{2m} (\nabla\theta)^2 n - \mathcal{E}(n, \nabla n) - U(\mathbf{r}) n, \quad (19)$$

where $n(\mathbf{r}, t)$ is the local density, $U(\mathbf{r})$ is the external potential acting on particles, and m the mass of superfluid particles. In the case of superfluid bosons $\theta(\mathbf{r}, t)$ is the phase of the condensate order parameter, while in the case of superfluid fermions $\theta(\mathbf{r}, t)$ is half of the phase of the condensate order parameter (of Cooper pairs). $\mathcal{E}(n, \nabla n)$ is the internal energy density of the system. Note that we are supposing that this equation of state $\mathcal{E}(n, \nabla n)$ can depend not only on the local density $n(\mathbf{r}, t)$ but also on its space derivatives. For this reason we call (19) the extended superfluid Lagrangian. We stress that in the context of the BCS-BEC crossover the extended internal energy density could be written as

$$\mathcal{E}(n, \nabla n) = \mathcal{E}_0(n) + \lambda \frac{\hbar^2}{8m} \frac{(\nabla n)^2}{n}, \quad (20)$$

where

$$\mathcal{E}_0(n) = \frac{3}{5} n \epsilon_F f\left(\frac{1}{k_F a}\right) \quad (21)$$

is the energy density discussed in the previous section (see Eq. (4)) which depends on the universal function $f(y)$ of the BCS-BEC crossover, while the second term is the gradient correction of the von-Weizsacker form [35]. In the BCS-BEC crossover we expect that $1/6 \leq \lambda \leq 1/4$, where $\lambda = 1/6$ is the appropriate value in BCS regime of weakly-interacting superfluid fermions of mass m [45, 46], while $\lambda = 1/4$ is the appropriate value in the deep BEC regime of weakly-interacting superfluid bosonic dimers of mass $2m$ [35].

By using the Lagrangian density (19) the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \nabla \cdot \frac{\partial \mathcal{L}}{\partial (\nabla \theta)} = 0 \quad (22)$$

gives

$$\frac{\partial n}{\partial t} + \frac{\hbar}{m} \nabla \cdot (n \nabla \theta) = 0. \quad (23)$$

The Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial n} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{n}} - \nabla \cdot \frac{\partial \mathcal{L}}{\partial (\nabla n)} = 0 \quad (24)$$

gives instead

$$\hbar \dot{\theta} + \frac{\hbar^2}{2m} (\nabla \theta)^2 + U(\mathbf{r}) + X(n, \nabla n) = 0, \quad (25)$$

where

$$X(n, \nabla n) = \frac{\partial \mathcal{E}}{\partial n} - \nabla \cdot \frac{\partial \mathcal{E}}{\partial (\nabla n)} \quad (26)$$

is the local chemical potential of the system (see also [54] and [55]).

The local field velocity $\mathbf{v}(\mathbf{r}, t)$ of the superfluid is related to the phase $\theta(\mathbf{r}, t)$ of the condensate by

$$\mathbf{v}(\mathbf{r}, t) = \frac{\hbar}{m} \nabla \theta(\mathbf{r}, t). \quad (27)$$

This definition ensures that the velocity is irrotational, i.e. $\nabla \wedge \mathbf{v} = \mathbf{0}$. By using the definition (27) in both Eqs. (23) and (25) and applying the gradient operator ∇ to Eq. (25) one finds the extended hydrodynamic equations of superfluids

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{v}) = 0 . \quad (28)$$

$$m \frac{\partial \mathbf{v}}{\partial t} + \nabla \left[\frac{1}{2} m \mathbf{v}^2 + U(\mathbf{r}) + X(n, \nabla n) \right] = \mathbf{0} . \quad (29)$$

We stress that in the presence of an external confinement $U(\mathbf{r})$ the chemical potential μ of the system does not coincide with the local chemical potential $X(n, \nabla n)$. The chemical potential μ can be obtained from Eq. (25) setting $\theta(\mathbf{r}, t) = -\mu t/\hbar$ and $\mathbf{v}(\mathbf{r}, t) = \mathbf{0}$, such that

$$U(\mathbf{r}) + X(n_0, \nabla n_0) = \mu , \quad (30)$$

where $n_0(\mathbf{r})$ is the ground-state local density.

The Lagrangian density (19) depends on the dynamical variables $\theta(\mathbf{r}, t)$ and $n(\mathbf{r}, t)$. The conjugate momenta of these dynamical variables are then given by

$$\pi_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = -\hbar n , \quad (31)$$

$$\pi_n = \frac{\partial \mathcal{L}}{\partial \dot{n}} = 0 , \quad (32)$$

and the corresponding Hamiltonian density reads

$$\mathcal{H} = \pi_\theta \dot{\theta} + \pi_n \dot{n} - \mathcal{L} = -\hbar n \dot{\theta} - \mathcal{L} , \quad (33)$$

namely

$$\mathcal{H} = \frac{\hbar^2}{2m} (\nabla \theta)^2 n + \mathcal{E}(n, \nabla n) + U(\mathbf{r}) n , \quad (34)$$

which is the sum of the flow kinetic energy density $\hbar^2 (\nabla \theta)^2 n / (2m) = (1/2) m v^2 n$, the internal energy density $\mathcal{E}(n, \nabla n)$, and the external energy density $U(\mathbf{r}) n$.

2.1 Extended hydrodynamics in terms of Goldstone field

Note that taking into account Eq. (25) one immediately finds

$$X(n, \nabla n) n = -\hbar \dot{\theta} n - \frac{\hbar^2}{2m} (\nabla \theta)^2 n - U(\mathbf{r}) n . \quad (35)$$

Consequently the Lagrangian density (19) can be rewritten as

$$\mathcal{L} = X(n, \nabla n) n - \mathcal{E}(n, \nabla n) . \quad (36)$$

Remarkably

$$P(n, \nabla n) = X(n, \nabla n) n - \mathcal{E}(n, \nabla n) \quad (37)$$

is the local pressure of the system as a function of the density and its spatial derivatives, which can be written as a function of the local chemical potential X and its spatial derivatives, namely

$$\mathcal{L} = P(X, \nabla X) . \quad (38)$$

This result, based on a Legendre transformation, is clearly illustrated in the book of Popov [56] and used in the recent papers of Son and Wingate [54] and Manes and Valle [55]. Finally one can introduce the Goldstone field $\phi(\mathbf{r}, t)$ as

$$\phi(\mathbf{r}, t) = \theta(\mathbf{r}, t) + \frac{\mu}{\hbar} t . \quad (39)$$

In this way, by using again Eq. (25), one can write

$$X = \mu - \hbar \dot{\phi} - \frac{\hbar^2}{2m} (\nabla \phi)^2 - U(\mathbf{r}) , \quad (40)$$

Thus, the Lagrangian density (38) actually depends only on the Goldstone field $\phi(\mathbf{r}, t)$. This is exactly the main message of the paper of Son and Wingate [54], which however traces back to the older results of Popov [56].

2.2 Application: the Unitary Fermi gas

Let us now suppose that the equation of state of the superfluid at zero temperature is that of the unitary Fermi gas, i.e.

$$\mathcal{E}(n, \nabla n) = \frac{3}{5} \frac{\hbar^2}{2m} (3\pi^2)^{2/3} \xi n^{5/3} + \lambda \frac{\hbar^2}{8m} \frac{(\nabla n)^2}{n} , \quad (41)$$

where $\xi \simeq 0.4$ and $\lambda \simeq 0.25$ [37, 41]. It follows from Eq. (26) that

$$X(n, \nabla n) = \frac{\hbar^2}{2m} (3\pi^2)^{2/3} \xi n^{2/3} - \lambda \frac{\hbar^2}{8m} \frac{(\nabla n)^2}{n^2} , \quad (42)$$

by taking into account that the surface terms give zero contribution. In addition we get from Eq. (37) that

$$P(n, \nabla n) = \frac{2}{5} \frac{\hbar^2}{2m} (3\pi^2)^{2/3} \xi n^{5/3} - \lambda \frac{\hbar^2}{4m} \frac{(\nabla n)^2}{n} , \quad (43)$$

The Lagrangian density of Eq. (38) is then obtained by finding n and ∇n as functions of X and ∇X by inverting Eq. (42). This can be done in terms of a derivative expansion. One gets $n = (2m\xi)^{3/2} X^{3/2} / (3\pi^2 \hbar^3)$ and

$$\mathcal{L} = \mathcal{L}_{LO} + \mathcal{L}_{NLO} \quad (44)$$

where

$$\mathcal{L}_{LO} = c_0 \frac{m^{3/2}}{\hbar^3} X^{5/2} , \quad (45)$$

with

$$c_0 = \frac{2^{5/2}}{15\pi^2 \xi^{3/2}} , \quad (46)$$

is the Lagrangian density at the leading order, and

$$\mathcal{L}_{NLO} = c_1 \frac{m^{1/2}}{\hbar} \frac{(\nabla X)^2}{\sqrt{X}} , \quad (47)$$

with

$$c_1 = -\lambda \frac{3 \cdot 2^{1/2}}{8\pi^2 \xi^{3/2}} , \quad (48)$$

is the next-to-leading contribution to the Lagrangian density. The Lagrangian density (44) is the same of that derived by Son and Wingate [54] from general coordinate invariance and conformal invariance. Actually, at the next-to-leading order Son and Wingate have found an additional term [54], which has been questioned by Manes and Valle [55] and is absent in our approach.

2.3 Nonlinear sound waves, static response function and structure factor

In this subsection we consider the following zero-temperature equation of state of a generic superfluid

$$\mathcal{E}(n, \nabla n) = \mathcal{E}_0(n) + \lambda \frac{\hbar^2}{2m} \frac{(\nabla n)^2}{4n} . \quad (49)$$

Here the internal energy is the sum of two contributions: a generic internal energy $\mathcal{E}_0(n)$ which depends only of the local density $n(\mathbf{r}, t)$ (for instance that of Eq. (21)) plus the gradient correction of the von Weizsäcker type, where the coefficient λ can be a function of the interaction strength.

The equation of motion (29) becomes

$$m \frac{\partial \mathbf{v}}{\partial t} + \nabla \left[\frac{1}{2} m \mathbf{v}^2 + U(\mathbf{r}) + X_0(n) - \lambda \frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{n}}{\sqrt{n}} \right] = \mathbf{0} , \quad (50)$$

with

$$X_0(n) = \frac{\partial \mathcal{E}_0}{\partial n} . \quad (51)$$

We are interested on the propagation of sound waves in superfluids. For simplicity we set

$$U(\mathbf{r}) = 0 , \quad (52)$$

and consider a small perturbation $\tilde{n}(\mathbf{r}, t)$ around a uniform and constant configuration n_0 , namely

$$n(\mathbf{r}, t) = n_0 + \tilde{n}(\mathbf{r}, t) . \quad (53)$$

Neglecting quadratic terms in \tilde{n} and v we derive the linearized equations of extended superfluid hydrodynamics

$$\frac{\partial \tilde{n}}{\partial t} + n_0 \nabla \cdot \mathbf{v} = 0 , \quad (54)$$

$$n_0 \frac{\partial \mathbf{v}}{\partial t} + c_s^2 \nabla \tilde{n} - \lambda \frac{\hbar^2}{4m^2} \nabla (\nabla^2 \tilde{n}) = \mathbf{0} , \quad (55)$$

where c_s is the sound velocity, given by

$$c_s^2 = \frac{n_0}{m} \frac{\partial X_0(n_0)}{\partial n} = \frac{n_0}{m} \frac{\partial^2 \mathcal{E}_0(n_0)}{\partial n^2} . \quad (56)$$

Applying the operator $\frac{\partial}{\partial t}$ to Eq. (54) and the operator ∇ to Eq. (55) and subtracting the two resulting equations we get

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2 + \lambda \frac{\hbar^2}{4m^2} \nabla^4 \right) \tilde{n}(\mathbf{r}, t) = 0 . \quad (57)$$

This is the wave equation of the small perturbation $\tilde{n}(\mathbf{r}, t)$ around the uniform density n_0 . We stress that the effect of the gradient term in the equation of state is the presence of quartic spatial derivatives in this wave equation.

It is straightforward to show that the wave equation admits the real solution

$$\tilde{n}(\mathbf{r}, t) = A e^{i(\mathbf{q} \cdot \mathbf{r} - \omega_q t)} + A^* e^{-i(\mathbf{q} \cdot \mathbf{r} - \omega_q t)} , \quad (58)$$

where the frequency ω_q and the wave vector \mathbf{q} are related by the Bogoliubov-like dispersion formula

$$\hbar\omega_q = \sqrt{\frac{\hbar^2 q^2}{2m} \left(\lambda \frac{\hbar^2 q^2}{2m} + 2mc_s^2 \right)}, \quad (59)$$

or equivalently

$$\omega_q = c_s q \sqrt{1 + \alpha q^2} = c_s q \left(1 + \frac{\alpha}{2} q^2 - \frac{\alpha^2}{8} q^4 + \dots \right), \quad (60)$$

with $\alpha = \lambda \hbar^2 / (4m^2 c_s^2)$. Thus, the dispersion relation ω_q is linear in $q = |\mathbf{q}|$ only for small values of the wavenumber q and becomes quadratic for large values of q .

We observe that, for a generic many-body system, the dispersion relation can be written as [57]

$$\hbar\omega_q = \sqrt{\frac{m_1(q)}{m_{-1}(q)}}, \quad (61)$$

where $m_n(q)$ is the n moment of the dynamic structure function $S(q, \omega)$ of the many-body system under investigation, i.e.

$$m_n(q) = \int_0^\infty d\omega S(q, \omega) (\hbar\omega)^n. \quad (62)$$

Note that Eq. (61) is not exact and is valid under the approximation of a single-mode density excitation. It therefore only gives an upper bound for the dispersion relation. This is important for large q , for which quasiparticles other than density excitations may contribute. In our problem we have

$$m_1(q) = \frac{\hbar^2 q^2}{2m} \quad (63)$$

and

$$m_{-1}(q) = \frac{1}{\lambda \frac{\hbar^2 q^2}{2m} + 2mc_s^2}. \quad (64)$$

In general, the static response function $\chi(q)$ is defined as [57]

$$\chi(q) = -2 m_{-1}(q), \quad (65)$$

and in our problem it reads

$$\chi(q) = -\frac{2}{\lambda \frac{\hbar^2 q^2}{2m} + 2mc_s^2}, \quad (66)$$

or equivalently

$$\chi(q) = -\frac{1}{mc_s^2} \frac{1}{1 + \alpha q^2} = -\frac{1}{mc_s^2} (1 - \alpha q^2 + \alpha^2 q^4 + \dots), \quad (67)$$

where again $\alpha = \lambda \hbar^2 / (4m^2 c_s^2)$.

The static structure factor $S(q)$ is instead defined as [57]

$$S(q) = m_0(q) = \int_0^\infty d\omega S(q, \omega), \quad (68)$$

but it can be approximated by the expression

$$\tilde{S}(q) = \sqrt{m_1(q) m_{-1}(q)}, \quad (69)$$

which gives an upper bound of $S(q)$, i.e. $\tilde{S}(q) \geq S(q)$ [57]. In our problem we immediately find

$$\tilde{S}(q) = \sqrt{\frac{\frac{\hbar^2 q^2}{2m}}{\lambda \frac{\hbar^2 q^2}{2m} + 2mc_s^2}}, \quad (70)$$

or equivalently

$$\tilde{S}(q) = \frac{\hbar q}{2mc_s} \frac{1}{\sqrt{1 + \alpha q^2}} = \frac{\hbar q}{2mc_s} \left(1 - \frac{1}{2} \alpha q^2 + \frac{3}{8} \alpha^2 q^4 + \dots \right). \quad (71)$$

Our results clearly indicate that it should be possible to observe experimentally the effect of the dispersive von-Weizsacker-like gradient term from sound-wave measurements.

3 Conclusions

In the first part of this contribution we have calculated the contact C as a function of the inverse scattering parameter $1/(k_F a)$ for a uniform superfluid Fermi gas in the full BCS-BEC crossover at zero temperature. We have found that the contact C has a maximum close to the unitarity limit of infinite scattering length, in analogy with the behavior of the Landau's critical velocity v_c , at which there is the breaking of superfluid motion [33]. We have also considered the interacting Fermi system under harmonic confinement. In this case, we have derived analytical formulas of the contact intensity C in the three relevant limits of the crossover. Our results can be experimentally tested with ultracold atomic clouds by measuring one of the quantities which are directly related to the contact intensity C : the tail of the momentum distribution, the derivative of the total energy with respect to the scattering length, the radio-frequency spectroscopy shift, or the photoassociation rate. In the second part we have analyzed some properties of the extended superfluid hydrodynamics [34]; in particular, we have shown its strict relation with the low-energy effective field theory built on the Goldstone mode. Finally, by using the extended hydrodynamics we have calculated, for generic superfluid in the absence of external confinement, the nonlinear dispersion relation of sound waves, and, as a by-product, both static response function and structure factor.

References

1. S. Tan, Ann. Phys. **323**, 2952 (2008).
2. S. Tan, Ann. Phys. **323**, 2971 (2008).
3. S. Tan, Ann. Phys. **323**, 2987 (2008).
4. S. Zhang and A.J. Leggett, Phys. Rev. A **79** 023601 (2009).
5. M. Punk and W. Zwerger, Phys. Rev. Lett. **99** 170404 (2007).

6. G. Baym, C.J. Pethick, Z. Yu, and M.W. Zwierlein, Phys. Rev. Lett. **99**, 190407 (2007).
7. P. Pieri, A. Perali, and G.C. Strinati, Nature Physics **5**, 736 (2009).
8. J.P. Gaebler, J.T. Stewart, T.E. Drake, D.S. Jin, A. Perali, P. Pieri, G.C. Strinati, Nature Physics **6**, 569 (2010).
9. F. Werner, L. Tarruell, and Y. Castin, Eur. Phys. J. B **68**, 401 (2009).
10. E. Braaten, in *The BCS-BEC Crossover and the Unitary Fermi Gas*, edited by W. Zwerger (Springer, Berlin, 2012).
11. E. Braaten and L. Platter, Phys. Rev. Lett. **100**, 205301 (2008).
12. E. Braaten, D. Kang, and L. Platter, Phys. Rev. A **78**, 053606 (2008).
13. E. Braaten and L. Platter, Laser Phys. **19**, 550 (2009).
14. A. Bulgac, J.E. Drut, and P. Magierski, Phys. Rev. Lett. **96**, 090404 (2006).
15. A. Bulgac, J.E. Drut, and P. Magierski, Phys. Rev. Lett. **99**, 120401 (2007).
16. K. Huang and C.N. Yang, Phys. Rev. **105** 767 (1957).
17. T.D. Lee and C.N. Yang, Phys. Rev. **105** 1119 (1957).
18. G.E. Astrakharchik, J. Boronat, J. Casulleras, and S. Giorgini, Phys. Rev. Lett. **93**, 200404 (2004).
19. J. Carlson, S. Gandolfi, K.E. Schmidt, and S. Zhang, Phys. Rev. A **84**, 061602(R) (2011).
20. N. Navon, S. Nascimbene, F. Chevy, and C. Salomon, Science **328**, 729 (2010).
21. A. Bulgac and G.F. Bertsch, Phys. Rev. Lett. **94**, 070401 (2005).
22. D.S. Petrov, C. Salomon, and G.V. Shlyapnikov, Phys. Rev. Lett. **93**, 090404 (2004).
23. T.D. Lee, K. Huang, and C.N. Yang, Phys. Rev. **106**, 1135 (1957).
24. Z. Yu, G.M. Bruun, and G. Baym, Phys. Rev. A **80**, 023615 (2009).
25. N. Manini and L. Salasnich, Phys. Rev. A, **71**, 033625 (2005).
26. G. Diana, N. Manini, and L. Salasnich, Phys. Rev. A, **73**, 065601 (2006).
27. T.N. De Silva and E.J. Mueller, Phys. Rev. A **72**, 063614 (2005).
28. Yu. Zhou and G. Huang, Phys. Rev. A **75**, 023611 (2007).
29. T.K. Ghosh, Phys. Rev. A **76**, 033602 (2007).
30. S.K. Adhikari, Phys. Rev. A **77**, 045602 (2008).
31. S.K. Adhikari, Phys. Rev. A **79**, 023611 (2009).
32. Y.E. Kim and A.L. Zubarev, Phys. Rev. A **70**, 033612 (2004).
33. R. Combescot, M.Yu. Kagan and S. Stringari, Phys. Rev. A **74**, 042717 (2006).
34. L. Salasnich and F. Toigo, Phys. Rev. A **78**, 053626 (2008).
35. L. Salasnich, Laser Phys. **19**, 642 (2009).
36. F. Ancilotto, L. Salasnich, and F. Toigo, Phys. Rev. A **79**, 033627 (2009).
37. S.K. Adhikari and L. Salasnich, New J. Phys. **11**, 023011 (2009).
38. L. Salasnich, F. Ancilotto, and F. Toigo, Laser Phys. Lett. **7**, 78 (2010).
39. L. Salasnich, EPL **96**, 40007 (2011).
40. F. Ancilotto, L. Salasnich, and F. Toigo, Phys. Rev. A **85**, 063612 (2012).
41. L. Salasnich, Few-Body Syst. DOI 10.1007/s00601-012-0442-y.
42. L. Salasnich, J. Math. Phys. **41**, 8016 (2000).
43. F. Dalfovo, S. Giorgini, L.P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).
44. C.F. von Weizsäcker, Zeit. Phys. **96**, 431 (1935).
45. N.H. March and M. P. Tosi, Proc. R. Soc. A **330**, 373 (1972).
46. E. Zaremba and H.C. Tso, Phys. Rev. B **49**, 8147 (1994).
47. M.A. Escobedo, M. Mannarelli and C. Manuel, Phys. Rev. A **79**, 063623 (2009).

- 48. E. Lundh and A. Cetoli, Phys. Rev. A **80**, 023610 (2009).
- 49. G. Rupak and T. Schäfer, Nucl. Phys. A **816**, 52 (2009).
- 50. S.K. Adhikari, Laser Phys. Lett. **6**, 901 (2009).
- 51. W.Y. Zhang, L. Zhou, and Y.L. Ma, EPL **88**, 40001 (2009).
- 52. A. Csordas, O. Almasy, and P. Szepfalusy, Phys. Rev. A **82**, 063609 (2010).
- 53. S. N. Klimin, J. Tempere, and J.P.A. Devreese, J. Low Temp. Phys. **165**, 261 (2011).
- 54. D.T. Son and M. Wingate, Ann. Phys. **321**, 197 (2006).
- 55. J.L. Manes and M.A. Valle, Ann. Phys. **324**, 1136 (2009).
- 56. V.N. Popov, *Functional Integrals in Quantum Field Theory and Statistical Physics* (Reidel, Dordrecht, 1983).
- 57. F. Dalfovo, A. Latri, L. Pricapenko, S. Stringari, and J. Treiner, Phys. Rev. B **52**, 1193 (1995).